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Study Material

16/3/20

Topic :- Improper Riemann Integrals

Improper Integral of Type I Let $a < b$ and f be integrable on $[c, b]$ $\forall$ $a < c < b$ but f is not integrable on $[a, b]$. Then $\int_a^b f$ is called an Improper integral of type I.

If $\lim_{c \to a^+} \int_c^b f$ exists then Improper integral converges and we write $\int_a^b f = \lim_{c \to a^+} \int_c^b f$.

If this limit does not exist, then the Improper integral diverges.

Improper Integral of Type II If $a < b$ and f be integrable on $[a, c]$ $\forall$ $a < c < b$ but f is not integral on $[a, b]$. Then $\int_a^b f$ is called Improper Integral of type II.

If $\lim_{c \to b^-} \int_a^c f$ exists then the Improper integral Converges and we write

$$\int_a^b f = \lim_{c \to b^-} \int_a^c f .$$

<u>Example 7.8.4</u> Show that each of the following is an Improper Integral & determine their Convergence or divergence.

(a) $\displaystyle\int_0^1 \frac{1}{x^2}\, dx$ (b) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\, dx$.

<u>Solⁿ</u> (a) Let $f(x) := \frac{1}{x^2}$

Clearly, f is Continuous & bdd on any closed interval $[c, 1]$ for $0 < c < 1$. Thus $f \in R[c,1]$

but f is not defined at $x = 0$.

Thus $\displaystyle\int_0^1 \frac{1}{x^2}\, dx$ is Improper.

Now, for $0 < c < 1$, $\displaystyle\int_c^1 \frac{1}{x^2}\, dx = \left(-\frac{1}{x}\right)_c^1$

$$= \frac{1}{c} - 1$$

Thus $\lim_{c \to 0^+} \int_c^1 \frac{1}{x^2}\,dx = \infty$.

Therefore $\int_0^1 \frac{1}{x^2}\,dx$ diverges.

(b) Let $f(x) := \frac{1}{\sqrt{x}}$, f is continuous, bdd on some $[c,1]$, $0 < c < 1$. Thus $f \in R[c,1]$.

but f is not defined at $x = 0$.

Thus $\int_0^1 \frac{1}{\sqrt{x}}$ is improper one.

Now for $0 < c < 1$, $\int_c^1 \frac{1}{\sqrt{x}}\,dx = (2\sqrt{x})_1^c$

$= 2(1 - \sqrt{c})$

$\lim_{c \to 0^+} \int_c^1 \frac{1}{\sqrt{x}}\,dx = 2$

So $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ cgs.

Thm 7.8.5 (Comparision Test I) Let $a < b$, and $f, g \in R[a, x]$.

$\forall\ a < x < b$, $0 \le f(x) \le g(x)$

If $\int_a^b g$ converges so $\int_a^b f$ converges.

and $\int_a^b f \le \int_a^b g$.

<u>Proof.</u> Let $G(x) := \int_a^x g(t)\,dt$, $a < x < b$

$\Rightarrow$ $G'(x) = g(x)$ (by fundamental Thm of Calculus)

Given $g(x) \geq 0$ $\therefore$ $G'(x) \geq 0$. Thus G is monotone $\uparrow$ on $[a, b)$. Since $\int_a^b g$ converges

and $\lim_{x \to b^-} \int_a^b g$ exists, so

$$\int_a^b g = \lim_{x \to b^-} \int_a^x g = \sup \left\{ \int_a^x g : a < x < b \right\}$$

$$\Rightarrow \int_a^x g \le \int_a^b g \quad \forall\, x \in [a, b).$$

Also, $f(x) \le g(x)$ $\Rightarrow$ $\int_a^x f \le \int_a^x g \le \int_a^b g$

Therefore $\lim_{x \to b^-} \int_a^x f$ exists and is $\le \int_a^b g$.

i.e. $\int_a^b f \le \int_a^b g$.

<u>Ex 7.8.7</u> Show that $\int_0^1 \frac{1}{x^2 + x}\,dx$ cgs.

Sol^n Let $f(x) := \dfrac{1}{x^2+x}$

$$g(x) = \frac{1}{x^2}$$

$$\Rightarrow \quad 0 \le f(x) \le g(x)$$

Since $\int_0^1 \frac{1}{x^2}$ Cgs $\forall x \in (0,1]$

$\therefore$ By Comparison test, $\int_0^1 \frac{1}{x^2+x}$ Cgs.

Thm 7.8.8 (Absolute Convergence) Let $\int_a^b f$ be an Improper integral of type I. If $\int_a^b |f|$ converges so $\int_a^b f$ Converges.

Proof. Let $\int_a^b f$ be Improper Integral of Type I. For all $x \in (a, b]$,

$$-|f(x)| \le f(x) \le |f(x)|$$

$$\Rightarrow \quad 0 \le f(x) + |f(x)| \le 2|f(x)|$$

But $\int_a^b |f|$ Cgs, so $\int_a^b 2|f|$ Cgs. Hence by Comparison test, $\int_a^b f + |f|$ Converges.

Now $\forall$ $a < c < b$, $\int_c^b f = \int_c^b (f + |f| - |f|)$

$$\Rightarrow \int_c^b f = \int_c^b (f+|f|) - \int_c^b |f|$$

Thus $\lim_{c \to a^+} \int_c^b f$ exists $\Rightarrow \int_a^b f$ cgs.

Exercise Set 7.8 A

Q3 Check whether $\int_a^b f$ is Improper Integral and also determine its convergence or divergence

(a) $\int_0^1 \frac{1}{x}\, dx$

Sol^n Let $f(x) := \frac{1}{x}$, $f \in R[c,1]$ $\forall\ 0<c<1$.

$$\Rightarrow \int_c^1 \frac{1}{x}\, dx = \log 1 - \log c = -\log(c)$$

$$\Rightarrow \lim_{c \to 0^+} \int_c^1 \frac{1}{x}\, dx = -\infty$$

Thus $\int_0^1 \frac{1}{x}\, dx$ dgs.

(b) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

(Try yourself)

(c) $\int_0^1 \frac{1}{e^x}\,dx$

Sol^n Let $f(x) = \frac{1}{e^x}$, $f \in R[0;1]$

and $\int_0^1 \frac{1}{e^x}\,dx = (-e^{-x})_0^1 = 1 - e^{-1} < \infty$

This $\int_0^1 \frac{1}{e^x}\,dx$ is proper Integral.

(d) $\int_1^2 \frac{1}{x\ln(x)}\,dx$ (Try Yourself)